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AN ASYMPTOTIC ANALYSIS OF END EFFECTS IN THE AXISYMMETRIC DEFORMATION OF ELASTIC TUBES WEAK IN SHEAR: HIGHER-ORDER SHELL THEORIES ARE INADEQUATE AND UNNECESSARY†

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Abstract—This paper specializes to a semi-infinite tube Horgan's (*Int. J. Solids Structures* **10**, 837–852, 1974) two-stress-function formulation of the equations for the axisymmetric deformation of a linearly elastic transversely isotropic cylindrical body free of surface tractions. The ratio of the tube's shear modulus to its radial (transverse) extensional modulus is taken to be of the order of magnitude of the square root of its thickness to its mean radius. The equations are solved by formal asymptotic expansion in (fractional) powers of the thickness to radius ratio for four canonical sets of end conditions: (A) axisymmetric equilibrated tractions; (B) and (C), two different combinations of tractions and displacements; and (D) axisymmetric radial and axial displacements. The solutions exhibit interior (i.e. shell-like) parts and wide and narrow boundary (or edge) layers, the latter containing components that vary extremely rapidly through the thickness of the tube. The analysis focuses on computing the lowest-order correction, both in the interior and in the boundary layers, to classical shell theory. It is shown that in cases (A)–(C) the interior correction to classical shell theory—that is, those effects so-called higher-order shell theories attempt to capture—can (ultimately) be determined directly, in terms of the edge data, but that in case of prescribed displacements (D), the computation of (three-dimensional) boundary-layer effects is essential. These conclusions are consistent with those for elastically *isotropic* shells found by Gregory and Wan (1992) who used ingenious arguments based on the Betti Reciprocity Principle.

1. INTRODUCTION

My aim is to substantiate the title of this paper by a formal asymptotic analysis of the exact equations of linear elasticity for a semi-infinite, transversely isotropic *tube* (i.e. a circular cylindrical shell) of inner radius $R-H$ and outer radius $R+H$. I start from the formulation of Horgan (1974) in which the stresses, σ_r, σ_θ, σ_z, and τ, and the radial and axial displacements, u_r and u_z, are expressed in terms of two functions, ϕ and χ, that satisfy two coupled, second-order partial differential equations in the cylindrical coordinates r and z. The sides of the tube, $r = R \pm H$, are traction free while the end, $z = 0$, is subjected to one of four sets of conditions [the classification is that of Gregory and Wan (1992)]:

(A) axisymmetric, self-equilibrating tractions

$$\sigma_z(r,0) = \hat{\sigma}_z(r), \quad \tau(r,0) = \hat{\tau}(r); \tag{1}$$

(B) axisymmetric self-equilibrated axial traction and radial displacement

$$\sigma_z(r,0) = \hat{\sigma}_z(r), \quad u_r(r,0) = \hat{u}_r(r); \tag{2}$$

(C) axisymmetric radial traction and axial displacement

†To Lyell Sanders on his 65th birthday in appreciation of his friendship and in admiration of his profound contributions to shell theory.

$$\tau(r,0) = \hat{\tau}(r), \quad u_z(r,0) = \hat{u}_z(r); \tag{3}$$

(D) axisymmetric displacements

$$u_z(r,0) = \hat{u}_z(r), \quad u_r(r,0) = \hat{u}_r(r), \tag{4}$$

where a hat ($\hat{}$) denotes a prescribed quantity. Any rigid body displacements are to be suppressed so that, in each case, the stresses and displacements in the tube decay to zero as $z \to \infty$. As will be seen, each case is more complicated than its predecessor.

In a linearly elastic transversely isotropic tube undergoing axisymmetric deformation, there are five elastic constants. These may be taken as the coefficients (or their reciprocals) which appear when twice the strain-energy density is written as a quadratic function of the stresses, as follows (Horgan, 1974):

$$2V = \frac{1}{E_T}(\sigma_r^2+\sigma_\theta^2) + \frac{1}{E_L}\sigma_z^2 + \frac{1}{\mu}\tau^2 - \frac{2\bar{\nu}}{E_T}\sigma_z(\sigma_r+\sigma_\theta) - \frac{2\nu_{TT}}{E_T}\sigma_r\sigma_\theta. \tag{5}$$

By *weak in shear* I mean that $\mu/E_T = O[(H/R)^{1/2}]$. With this characterization I shall show, formally, that as $H/R \to 0$, the (unique) solution of the governing equations exhibits *three* distinct boundary (or edge) layers: the well-known *shell layer* of width $O(H^{1/2}R^{1/2})$; a *wide layer* of width $O[(R/H)^{1/4}H]$ (i.e. wide relative to the tube's thickness); and a *narrow layer* of width $O[(H/R)^{1/4}H]$. These latter two boundary layers, analogous to the "wide" and "narrow" boundary layers found by Horgan and Simmonds (1991) in a transversely isotropic elastic strip weak in shear, give rise to the end effects mentioned in the title. Moreover, the narrow layer contains a *sinuous* part that oscillates rapidly through the thickness of the tube.

An analysis of the boundary conditions at the end of the tube in cases (A)–(D) leads to the conclusion that *if displacements are prescribed at the end of the tube [Case (D)] then proper corrections to the classical shell equations due to transverse shear effects—a major aim of so-called higher-order shell theories—cannot be made without a consideration of three-dimensional edge effects associated with the wide and narrow boundary layers.* Thus, higher-order shell theories based on refined thickness distributions of stress and displacement in the interior of the shell cannot always be correct asymptotically, and in this sense are *inadequate*. Higher-order shell theories are also *unnecessary* in that a first correction to classical shell theory in the interior can be obtained by a simple iteration of the classical shell solution, involving no more than the integration of polynomials through the thickness. Determining the proper edge conditions for these iterated solutions requires, as I shall show, an analysis of edge effects, *although, in cases (A)–(C), edge conditions for the iterated interior solutions can be expressed ultimately in terms of weighted integrals of the edge data.* Gregory has suggested to me that a simpler (and more fundamental) way to obtain boundary conditions for the interior (shell-like) solutions in cases (A)–(C) is to use the Betti Reciprocity Principle, as Gregory and Wan (1992) did for isotropic tubes. This interesting alternative remains to be examined.

The analysis herein could have been developed by first expressing the solutions of the governing equations in terms of Bessel functions (Warren *et al.*, 1967) and then using ideas analogous to those of Gregory and Wan (1992) to decompose the edge data into "shell-like" parts and residual "local self-equilibrating" parts. The algebraic details of such an exact approach are tedious. In contrast, *the present direct asymptotic solution of the governing differential equations is immensely simpler.* (At one point in their analysis of *isotropic* tubes, Gregory and Wan have to take 11 terms in an expansion of an exact solution to retain two non-zero terms in a later approximation.)

In Sections 3–6, where I develop asymptotic expansions for the interior, wide, narrow and sinuous solutions, I carry the expansions only far enough to compute the first-order correction to the classical shell solution (in the interior of the tube) and the lowest-order approximations to the wide and narrow boundary-layer solutions.

2. THE GOVERNING EQUATIONS

Let

$$A = \frac{(E_T/E_L)-\bar{\nu}^2}{1+\nu_{TT}} \quad \text{and} \quad B = \frac{E_T}{(1+\nu_{TT})\mu}-1-\bar{\nu} \tag{6}$$

by Horgan's (1974) auxiliary elastic constants and introduce dimensionless independent and dependent variables by setting

$$r = R+H\rho, \quad z = \sqrt{HR}\zeta, \qquad \phi = HR\sigma_0\Phi, \quad \chi = R^2\sqrt{HR}\sigma_0 X, \tag{7,8}$$

where σ_0 is some reference stress chosen to make the dominant stresses O(1). Then, in terms of the additional dimensionless parameters

$$\varepsilon \equiv \frac{H}{R}, \quad \kappa \equiv \frac{21}{1-\bar{\nu}}, \quad \delta = \frac{1-\bar{\nu}}{A}, \qquad \gamma \equiv \varepsilon^{1/2}[\nu_{TT}-1+(1-\bar{\nu})B/A], \tag{9}$$

eqns (3.17) and (3.18) of Horgan (1974) take the form

$$\Phi_{,\rho\rho}+\frac{\varepsilon\Phi_{,\rho}}{1+\varepsilon\rho}+\varepsilon(1-\nu_{TT})\kappa\Phi_{,\zeta\zeta} = \frac{\kappa X_{,\rho\zeta}}{1+\varepsilon\rho}, \tag{10}$$

$$X_{,\rho\rho}-\frac{\varepsilon X_{,\rho}}{1+\varepsilon\rho}+\varepsilon\delta X_{,\zeta\zeta} = -\varepsilon^{1/2}\gamma(1+\varepsilon\rho)\Phi_{,\rho\zeta}, \tag{11}$$

where a comma followed by a subscript denotes partial differentiation with respect to that subscript. The terminology *weak in shear*, defined following (5), means that $\gamma = O(1)$.

The stresses and displacements follow from eqns (3.12)–(3.16) of Horgan (1974) and from (7)–(9) above as

$$\frac{\sigma_r}{\sigma_0} = \Phi_{,\zeta\zeta}-\frac{X_{,\zeta}}{(1+\varepsilon\rho)^2}+\frac{\Phi_{,\rho}}{1+\varepsilon\rho}, \tag{12}$$

$$\frac{\sigma_\theta}{\sigma_0} = \bar{\nu}\varepsilon^{-1}\Phi_{,\rho\rho}+\frac{X_{,\zeta}}{(1+\varepsilon\rho)^2}+\nu_{TT}\Phi_{,\zeta\zeta}-\frac{(1-\bar{\nu})\Phi_{,\rho}}{1+\varepsilon\rho}, \tag{13}$$

$$\frac{r\sigma_z}{R\sigma_0} \equiv s(\rho,\zeta,\varepsilon) = \varepsilon^{-1}[(1+\varepsilon\rho)\Phi_{,\rho}]_{,\rho} = \varepsilon^{-1}\Phi_{,\rho\rho}+\rho\Phi_{,\rho\rho}+\Phi_{,\rho}, \tag{14}$$

$$\frac{R^{1/2}\tau}{H^{1/2}\sigma_0} \equiv t(\rho,\zeta,\varepsilon) = -\varepsilon^{-1}\Phi_{,\rho\zeta}, \tag{15}$$

$$\begin{aligned}\frac{2\bar{\mu}ru_z}{H^{1/2}R^{3/2}\sigma_0} &\equiv u(\rho,\zeta,\varepsilon)\\ &\equiv \Lambda(\zeta,\varepsilon)+\Psi(\zeta,\varepsilon)\rho+\varepsilon^{1/4}U(\rho,\zeta,\varepsilon)\\ &= \delta^{-1}[\varepsilon^{-1}X_{,\rho}-(1-\nu_{TT}+\bar{\nu}\delta)(1+\varepsilon\rho)\Phi_{,\zeta}],\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{2\bar{\mu}ru_r}{R^2\sigma_0} &\equiv w(\rho,\zeta,\varepsilon)\\ &\equiv \Delta(\zeta,\varepsilon)+\varepsilon W(\rho,\zeta,\varepsilon)\\ &= X_{,\zeta}-(1+\varepsilon\rho)\Phi_{,\rho}.\end{aligned} \tag{17}$$

The structure of the second lines of (16) and (17) emerges in the sections to follow from the asymptotic analysis of the governing differential equations and boundary conditions; Λ and Δ are thickness-averaged axial and radial displacements and Ψ is a thickness-averaged meridional rotation. To make the dimensionless *residual* axial and radial displacements, U and W, unique, I require that

$$\int_{-1}^{1} \{1, \rho\} U(\rho, \zeta, \varepsilon)\, \mathrm{d}\rho = 0 \quad \text{and} \quad \int_{-1}^{1} W(\rho, \zeta, \varepsilon)\, \mathrm{d}\rho = 0. \tag{18, 19}$$

The traction-free conditions on the cylindrical faces of the tube follow from (7), (8), and eqns (4.1) and (4.2) of Horgan (1974) as

$$\Phi_{,\rho}(\pm 1, \zeta, \varepsilon) = 0, \qquad (1 \pm \varepsilon)^2 \Phi_{,\zeta}(\pm 1, \zeta, \varepsilon) = X(\pm 1, \zeta, \varepsilon). \tag{20, 21}$$

The boundary conditions at $\zeta = 0$ for cases A–D discussed in the Introduction may be read off from (14)–(17). As $\zeta \to \infty$, Φ and X must approach zero. At the end of this section I shall reformulate the boundary conditions at $\zeta = 0$ in terms of Φ, $\Phi_{,\zeta}$, Λ, Ψ, U, Δ and W.

Finally, note that the *conservation property* of the solution ϕ given by eqn (4.11) of Horgan (1974), adapted to a tube and expressed in dimensionless variables, takes the form

$$(1 - \nu_{TT})\varepsilon \int_{-1}^{1} (1 + \varepsilon\rho)\Phi(\rho, \zeta, \varepsilon)\, \mathrm{d}\rho = [(1 + \varepsilon\rho)^2 \Phi(\rho, \zeta, \varepsilon)]_{-1}^{1}. \tag{22}$$

This relation, (16), (18), and the boundary condition (21) imply that

$$2\Lambda(\zeta, \varepsilon) = -\bar{\nu} \int_{-1}^{1} (1 + \varepsilon\rho)\Phi_{,\zeta}(\rho, \zeta, \varepsilon)\, \mathrm{d}\rho. \tag{23}$$

Because I require the axial displacement to approach zero as $\zeta \to \infty$, it turns out that $\Lambda(0, \varepsilon)$ *cannot* be prescribed as part of a displacement boundary condition. Rather, in all cases, Λ is to be determined by (23) or its equivalent.

In cases A–C (i.e. if σ_z or τ is prescribed at $z = 0$) I find it advantageous to work with alternative forms of the traction boundary conditions obtained by integrating (14) once and (15) twice with respect to ρ and using (20) and (22). Thus, with $\hat{s}(\rho, \varepsilon)$ denoting the prescribed value of $s(\rho, 0, \varepsilon)$, (14) and (20) imply that

$$\Phi(\rho, 0, \varepsilon) = \Phi(-1, 0, \varepsilon) + \varepsilon \hat{S}(\rho, \varepsilon), \tag{24}$$

where

$$\begin{aligned} \hat{S} &= \varepsilon^{-1} \int_{-1}^{\rho} \ln[(1 + \varepsilon\rho)/(1 + \varepsilon\bar{\rho})] \hat{s}(\bar{\rho}, \varepsilon)\, \mathrm{d}\bar{\rho} \\ &= \int_{-1}^{\rho} (\rho - \bar{\rho})\hat{s}_0(\bar{\rho})\, \mathrm{d}\bar{\rho} + \varepsilon \int_{-1}^{\rho} (\rho - \bar{\rho})[\hat{s}_4(\bar{\rho}) - (1/2)(\rho + \bar{\rho})\hat{s}_0(\bar{\rho})]\, \mathrm{d}\bar{\rho} + O(\varepsilon^2) \\ &\equiv \hat{S}_0(\rho) + \varepsilon \hat{S}_4(\rho) + O(\varepsilon^2). \end{aligned} \tag{25}$$

In arriving at the last line of (25), I have assumed—as I shall for all end data—that the prescribed dimensionless axial traction, $\hat{s}(\rho, \varepsilon)$, has an expansion in *integral* powers of ε. This is sufficient for the thesis of this paper, but, of course, a more general expansion is possible. On the other hand, as will be seen when the solutions of the various interior and boundary-layer equations are appropriately added and matched to the end conditions of the tube, all asymptotic expansions of *solutions* should proceed in powers of $\varepsilon^{1/4}$. The notation in the second and last lines of (25)—i.e. the subscript 4—is consistent with such expansions.

To express $\Phi(-1, 0, \varepsilon)$ in (24) in terms of $\hat{s}(\rho, \varepsilon)$, multiply (14) evaluated at $\zeta = 0$ by $(1-\nu_{TT})(1+\varepsilon\rho)^2$, integrate from -1 to 1, and invoke (20) and (22) to obtain

$$(1-\nu_{TT})\int_{-1}^{1}(1+\varepsilon\rho)^2\hat{s}(\rho,\varepsilon)\,d\rho = 2(1-\nu_{TT})$$
$$\times\left\{2\varepsilon\int_{-1}^{1}(1+\varepsilon\rho)\Phi(\rho,0,\varepsilon)\,d\rho-[(1+\varepsilon\rho)^2\Phi(\rho,0,\varepsilon)]_{-1}^{1}\right\} = 2(1+\nu_{TT})[(1+\varepsilon\rho)^2\Phi(\rho,0,\varepsilon)]_{-1}^{1}. \tag{26}$$

Then use (24) to write

$$\Phi(\rho,0,\varepsilon)|_{-1}^{1} = \varepsilon\hat{S}(1,\varepsilon) \tag{27}$$

and eliminate $\Phi(1, 0, \varepsilon)$ between this expression and (26) to get

$$\Phi(-1,0,\varepsilon) \equiv \hat{\Phi}(\varepsilon)$$
$$= \frac{(1+\varepsilon)^2}{4\varepsilon}\int_{-1}^{1}\left[\ln\left(\frac{1+\varepsilon\rho}{1+\varepsilon}\right)+\frac{(1-\nu_{TT})}{2(1+\nu_{TT})}\left(\frac{1+\varepsilon\rho}{1+\varepsilon}\right)^2\right]\hat{s}(\rho,\varepsilon)\,d\rho. \tag{28}$$

Using overall axial equilibrium,

$$\int_{-1}^{1}\hat{s}(\rho,\varepsilon)\,d\rho = 0, \tag{29}$$

and introducing the dimensionless stress couple at the edge,

$$\hat{m} \equiv \frac{\int_{R-H}^{R+H}(r-R)r\hat{\sigma}_z(r)\,dr}{H^2R\sigma_0} = \int_{-1}^{1}\rho\hat{s}(\rho,\varepsilon)\,d\rho \equiv \hat{m}_0+\varepsilon\hat{m}_4+O(\varepsilon^2), \tag{30}$$

I obtain the expansion

$$\hat{\Phi} = \frac{\hat{m}_0}{2(1+\nu_{TT})}+\frac{\varepsilon}{2}\left\{\hat{m}_0+\frac{1}{1+\nu_{TT}}\left[\hat{m}_4-\frac{\nu_{TT}}{2}\int_{-1}^{1}\rho^2\hat{s}_0(\rho)\,d\rho\right]\right\}+O(\varepsilon^2)$$
$$\equiv \hat{\Phi}_0+\varepsilon\hat{\Phi}_4+O(\varepsilon^2). \tag{31}$$

Now consider the integrated form of the edge condition $\tau(\rho,0) = \hat{\tau}(r)$. Upon integrating (15) from -1 to ρ, I find that

$$\Phi_{,\zeta}(\rho,0,\varepsilon) = \Phi_{,\zeta}(-1,0,\varepsilon)-\varepsilon\hat{T}(\rho,\varepsilon), \tag{32}$$

where

$$\hat{T} = \int_{-1}^{\rho}\hat{t}(\tilde{\rho},\varepsilon)\,d\tilde{\rho}$$
$$= \int_{-1}^{\rho}[\hat{t}_0(\tilde{\rho})+\varepsilon\hat{t}_4(\tilde{\rho})+O(\varepsilon^2)]\,d\tilde{\rho}$$
$$\equiv \hat{T}_0(\rho)+\varepsilon\hat{T}_4(\rho)+O(\varepsilon^2). \tag{33}$$

To express $\Phi_{,\zeta}(-1,0,\varepsilon)$ in (32) in terms of the dimensionless radial traction, $\hat{t}(\rho,\varepsilon)$, multiply (15) evaluated at $\zeta = 0$ by $\varepsilon(1-\nu_{TT})(1+\varepsilon\rho)^2$, integrate from -1 to 1, and invoke (22) to obtain

$$\varepsilon(1-\nu_{TT})\int_{-1}^{1}(1+\varepsilon\rho)^2\hat{t}(\rho,\varepsilon)\,d\rho = (1-\nu_{TT})$$
$$\times\left\{2\varepsilon\int_{-1}^{1}(1+\varepsilon\rho)\Phi_{,\zeta}(\rho,0,\varepsilon)\,d\rho-[(1+\varepsilon\rho)^2\Phi_{,\zeta}(\rho,0,\varepsilon)]_{-1}^{1}\right\} = (1+\nu_{TT})[(1+\varepsilon\rho)^2\Phi_{,\zeta}(\rho,0,\varepsilon)]_{-1}^{1}. \tag{34}$$

Then use (32) to write

$$[\Phi_{,\zeta}(\rho,0,\varepsilon)]_{-1}^{1} = -\varepsilon\hat{T}(1,\varepsilon) \tag{35}$$

and eliminate $\Phi_{,\zeta}(1,0,\varepsilon)$ between (34) and (35) to get

$$\Phi_{,\zeta}(-1,0,\varepsilon) \equiv \hat{\Phi}_{\zeta}(\varepsilon)$$
$$= \frac{(1+\varepsilon)^2}{4}\int_{-1}^{1}\left[1+\left(\frac{1-\nu_{TT}}{1+\nu_{TT}}\right)\left(\frac{1+\varepsilon\rho}{1+\varepsilon}\right)^2\right]\hat{t}(\rho,\varepsilon)\,d\rho. \tag{36}$$

If $\varepsilon = 1$ (a solid tube), this expression reduces to eqn (4.9) of Horgan (1974).

Introducing the dimensionless transverse shear stress resultant at the edge,

$$\hat{q} = \frac{\int_{R-H}^{R+H} r\hat{\tau}(r)\,dr}{H^{3/2}R^{1/2}\sigma_0} = \int_{-1}^{1}(1+\varepsilon\rho)\hat{t}(\rho,\varepsilon)\,d\rho \equiv \hat{q}_0+\varepsilon\hat{q}_4+O(\varepsilon^2), \tag{37}$$

I obtain the expansion

$$\hat{\Phi}_{\zeta} = \frac{1}{2(1+\nu_{TT})}\left\{\hat{q}_0+\varepsilon\left[(1+\nu_{TT})\hat{q}_0+\hat{q}_4-\nu_{TT}\int_{-1}^{1}\rho\hat{t}_0(\rho)\,d\rho\right]+O(\varepsilon^2)\right\}$$
$$\equiv \hat{\Phi}_{0\zeta}+\varepsilon\hat{\Phi}_{4\zeta}+O(\varepsilon^2). \tag{38}$$

Inserting (32) into (23) and noting (33) and (38), I obtain

$$2\Lambda(0,\varepsilon) = -\bar{\nu}\left[2\hat{\Phi}_{\zeta}(\varepsilon)-\varepsilon\int_{-1}^{1}(1+\varepsilon\rho)\hat{T}(\rho,\varepsilon)\,d\rho\right]$$
$$= -\frac{\bar{\nu}}{1+\nu_{TT}}\left\{\hat{q}_0+\varepsilon\left[\hat{q}_4+\int_{-1}^{1}\rho\hat{t}_0(\rho)\,d\rho\right]+O(\varepsilon^2)\right\}. \tag{39}$$

Using (14), (16) and (17) to introduce the notation

$$\hat{\Delta} \equiv \Delta(0,\varepsilon) = (1/2)\int_{-1}^{1}\hat{w}(\rho,\varepsilon)\,d\rho \equiv \hat{\Delta}_0+\varepsilon\hat{\Delta}_4+O(\varepsilon^2) \tag{40}$$

$$\varepsilon\hat{\Omega} \equiv \varepsilon W(\rho,0,\varepsilon)+(1+\varepsilon\rho)\Phi_{,\rho}(\rho,0,\varepsilon)$$
$$= \hat{w}(\rho,\varepsilon)-\hat{\Delta}(\varepsilon)+\varepsilon\int_{-1}^{\rho}\hat{s}(\tilde{\rho},\varepsilon)\,d\tilde{\rho}$$
$$\equiv \varepsilon[\hat{\Omega}_0(\rho)+\varepsilon\hat{\Omega}_4(\rho)+O(\varepsilon^2)], \tag{41}$$

$$\Lambda(0,\varepsilon) = (1/2)\int_{-1}^{1} \hat{u}(\rho,\varepsilon)\,\mathrm{d}\rho \equiv \Lambda_0 + \varepsilon^{1/2}\Lambda_2 + \varepsilon^{3/4}\Lambda_3 + \varepsilon\Lambda_4 + \mathrm{O}(\varepsilon^{5/4}), \tag{42†}$$

$$\hat{\Psi} = (3/2)\int_{-1}^{1} \rho\hat{u}(\rho,\varepsilon)\,\mathrm{d}\rho \equiv \hat{\Psi}_0 + \varepsilon\hat{\Psi}_4 + \mathrm{O}(\varepsilon^2), \tag{43}$$

$$\begin{aligned}\varepsilon^{1/4}\hat{U} &= \hat{u}(\rho,\varepsilon) - \hat{\Lambda}(\varepsilon) - \hat{\Psi}(\varepsilon)\rho \\ &\equiv \varepsilon^{1/4}[\hat{U}_0(\rho) + \varepsilon\hat{U}_4(\rho) + \mathrm{O}(\varepsilon^2)],\end{aligned} \tag{44}$$

and using (24), (32), and the first line of (39) in (50) below, I replace the boundary conditions (1)–(4) by:

Case A:

$$\Phi(\rho,0,\varepsilon) = \hat{\Phi}(\varepsilon) + \varepsilon\hat{S}(\rho,\varepsilon), \qquad \Phi_{,\zeta}(\rho,0,\varepsilon) = \hat{\Phi}_\zeta(\varepsilon) - \varepsilon\hat{T}(\rho,\varepsilon); \tag{45, 46}$$

Case B:

$$\Phi(\rho,0,\varepsilon) = \hat{\Phi}(\varepsilon) + \varepsilon\hat{S}(\rho,\varepsilon), \qquad X_{,\zeta}(\rho,0,\varepsilon) = \hat{\Delta}(\varepsilon) + \varepsilon\hat{\Omega}(\rho,\varepsilon); \tag{47, 48}$$

Case C:

$$\Phi_{,\zeta}(\rho,0,\varepsilon) = \hat{\Phi}_\zeta(\varepsilon) - \varepsilon\hat{T}(\rho,\varepsilon) \tag{49}$$

$$\begin{aligned}X_{,\rho}(\rho,0,\varepsilon) = \varepsilon\Big\{&\delta[\hat{\Psi}(\varepsilon)\rho + \varepsilon^{1/4}\hat{U}(\rho,\varepsilon)] + (1-\nu_{TT})(1+\varepsilon\rho)[\hat{\Phi}_\zeta(\varepsilon) - \varepsilon\hat{T}(\rho,\varepsilon)] \\ &+ \bar{\nu}\delta\varepsilon\Big[\rho\hat{\Phi}_\zeta(\varepsilon) + (1/2)\int_{-1}^{1}(1+\varepsilon\rho)\hat{T}(\rho,\varepsilon)\,\mathrm{d}\rho - (1+\varepsilon\rho)\hat{T}(\rho,\varepsilon)\Big]\Big\};\end{aligned} \tag{50}$$

Case D:

$$X_{,\rho}(\rho,0,\varepsilon) - \varepsilon(1-\nu_{TT}+\bar{\nu}\delta)(1+\varepsilon\rho)\Phi_{,\zeta}(\rho,0,\varepsilon) = \delta\varepsilon[\Lambda(\varepsilon) + \hat{\Psi}(\varepsilon)\rho + \varepsilon^{1/4}\hat{U}(\rho,\varepsilon)], \tag{51}$$

$$X_{,\zeta}(\rho,0,\varepsilon) - (1+\varepsilon\rho)\Phi_{,\rho}(\rho,0,\varepsilon) = \hat{\Delta}(\varepsilon) + \varepsilon\hat{W}(\rho,\varepsilon). \tag{52}$$

For use later, I note by (29), (30), (40), and the second line of (41) that

$$\int_{-1}^{1}\hat{\Omega}(\rho,\varepsilon)\,\mathrm{d}\rho = -\hat{m}(\varepsilon). \tag{53}$$

3. THE INTERIOR (SHELL-LIKE) EXPANSION

I first look for solutions to the differential equations (5) and (6) having asymptotic expansions of the form

† There are no "hats" on the Λs because, as noted following eqn (23), $\Lambda(0,\varepsilon)$ cannot be prescribed if one also requires that $\lim_{z\to\infty} u_z(r,z) = 0$. If $\Phi_{,\zeta}(\rho,0,\varepsilon)$ is prescribed, as in cases A and C—see (46) and (49)—then $\Lambda(0,\varepsilon)$ has an expansion in powers of ε. However, if $\Phi_{,\zeta}(\rho,0,\varepsilon)$ is not prescribed, as in cases B and D, then it turns out—see (124)—that (23) implies that $\Lambda(0,\varepsilon)$ must have the expansion indicated.

$$\Phi = F(\rho,\zeta,\varepsilon) = \overset{0}{F}(\rho,\zeta)+\varepsilon^{1/4}\overset{1}{F}(\rho,\zeta)+\varepsilon^{1/2}\overset{2}{F}(\rho,\zeta)+\cdots,$$
$$X = G(\rho,\zeta,\varepsilon) = \overset{0}{G}(\rho,\zeta)+\varepsilon^{1/4}\overset{1}{G}(\rho,\zeta)+\varepsilon^{1/2}\overset{2}{G}(\rho,\zeta)+\cdots, \tag{54}$$

such that derivatives of $\overset{n}{F}$ and $\overset{n}{G}$ with respect to ρ and ζ are O(1). Substituting (54) into (10) and (11) and the traction-free conditions (20) and (21), and equating coefficients of like power of $\varepsilon^{1/4}$, I obtain an infinite sequence of boundary-value problems free of ε. It turns out that $\overset{k}{F} = \overset{k}{G} = 0$, $k = 1, 3, 5$ so that the first four non-trivial sets of equations are

$$\overset{0}{F}_{,\rho\rho} = \kappa\overset{0}{G}_{,\rho\zeta}, \qquad \overset{0}{G}_{,\rho\rho} = 0, \qquad \overset{0}{F}_{,\rho}(\pm 1,\zeta) = 0, \qquad \overset{0}{F}_{,\zeta}(\pm 1,\zeta) = \overset{0}{G}(\pm 1,\zeta), \qquad (55)_0,(56)_0,(57)_0,(58)_0$$

$$\overset{2}{F}_{,\rho\rho} = \kappa\overset{2}{G}_{,\rho\zeta}, \qquad \overset{2}{G}_{,\rho\rho} = -\gamma\overset{0}{F}_{,\rho\zeta}, \qquad \overset{2}{F}_{,\rho}(\pm 1,\zeta) = 0, \qquad \overset{2}{F}_{,\zeta}(\pm 1,\zeta) = \overset{2}{G}(\pm 1,\zeta), \qquad (55)_2,(56)_2,(57)_2,(58)_2$$

$$\overset{4}{F}_{,\rho\rho} = \kappa\overset{4}{G}_{,\rho\zeta}-[\overset{0}{F}_{,\rho}+(1-\nu_{TT})\kappa\overset{0}{F}_{,\zeta\zeta}+\kappa\rho\overset{0}{G}_{,\rho\zeta}], \qquad \overset{4}{G}_{,\rho\rho} = -\gamma\overset{2}{F}_{,\rho\zeta}+\overset{0}{G}_{,\rho}-\delta\overset{0}{G}_{,\zeta\zeta}, \qquad (55)_4,(56)_4$$

$$\overset{4}{F}_{,\rho}(\pm 1,\zeta) = 0, \qquad \overset{4}{F}_{,\zeta}(\pm 1,\zeta)\pm 2\overset{0}{F}_{,\zeta}(\pm 1,\zeta) = \overset{4}{G}(\pm 1,\zeta). \qquad (57)_4,(58)_4$$

$$\overset{6}{F}_{,\rho\rho} = \kappa\overset{6}{G}_{,\rho\zeta}-[\overset{2}{F}_{,\rho}+(1-\nu_{TT})\kappa\overset{2}{F}_{,\zeta\zeta}+\kappa\rho\overset{2}{G}_{,\rho\zeta}], \qquad \overset{6}{G}_{,\rho\rho} = -\gamma(\overset{4}{F}_{,\rho}+\rho\overset{0}{F}_{,\rho})_{,\zeta}+\overset{2}{G}_{,\rho}-\delta\overset{2}{G}_{,\zeta\zeta}, \qquad (55)_6,(56)_6$$

$$\overset{6}{F}_{,\rho}(\pm 1,\zeta) = 0, \qquad \overset{6}{F}_{,\zeta}(\pm 1,\zeta)\pm 2\overset{2}{F}_{,\zeta}(\pm 1,\zeta) = \overset{6}{G}(\pm 1,\zeta). \qquad (57)_6,(58)_6$$

Solving $(56)_0$ and then $(55)_0$, I obtain

$$\overset{0}{G} = A_0(\zeta)+B_0(\zeta)\rho, \qquad \overset{0}{F} = C_0(\zeta)+D_0(\zeta)\rho+(1/2)\kappa B_0'(\zeta)\rho^2, \qquad (59)_0,(60)_0$$

where $A_0(\zeta)$, etc. are unknown functions. Imposition of $(57)_0$ and $(58)_0$ implies that $B_0 = D_0 = 0$, $A_0(\zeta) = C_0'(\zeta)$, but leaves C_0 undetermined. Thus,

$$\overset{0}{F} = C_0(\zeta), \qquad \overset{0}{G} = C_0'(\zeta). \qquad (61)_0$$

As $(55)_2$–$(58)_2$ differ in form from $(55)_0$–$(58)_0$ only in the right-hand side of $(56)_2$ which is now seen to be zero, it follows that

$$\overset{2}{F} = C_2(\zeta), \quad \overset{2}{G} = C_2'(\zeta), \qquad (61)_2$$

where $C_2(\zeta)$ is an unknown function.

Substituting $(61)_0$ and $(61)_2$ into $(56)_4$ and integrating, I obtain

$$\overset{2}{G} = A_4(\zeta)+B_4(\zeta)\rho-(1/2)\delta C_0^{(3)}(\zeta)\rho^2, \qquad (59)_4$$

and substituting this expression along with $(60)_0$ into $(55)_4$ and integrating, I obtain

$$\overset{4}{F} = C_4(\zeta)+D_4(\zeta)\rho+(1/2)\kappa[B_4(\zeta)-(1-\nu_{TT})C_0'(\zeta)]'\rho^2-(1/6)\delta\kappa C_0^{(4)}\rho^3. \qquad (60)_4$$

Here, $C_0^{(n)}(\zeta) \equiv d^nC_0(\zeta)/d\zeta^n$, $n \geqslant 3$, and $A_4(\zeta)$ etc. are unknown functions. The boundary condition $(57)_4$ implies that

$$B_4 = b_4 + (1-\nu_{TT})C'_0(\zeta), \quad D_4 = (1/2)\delta\kappa C_0^{(4)}(\zeta), \tag{62$_4$}$$

where b_4 is a constant that may be shown from (16) to produce an axial rigid body displacement and is thus discarded. The boundary condition $(58)_4$, the conservation property (22), and the relation $\kappa^{-1} = 1-\bar{\nu}$ coming from (9) imply

$$A_4 = (1/2)\delta C_0^{(3)}(\zeta) + C'_4(\zeta) \tag{63$_4$}$$

and the well-known governing differential equation of *classical* shell theory,

$$\delta C_0^{(4)}(\zeta) + 3(1-\bar{\nu})(1+\nu_{TT})C_0(\zeta) = 0. \tag{64$_4$}$$

Thus,

$$\overset{4}{G} = C'_4(\zeta) + (1/2)\delta C_0^{(3)}(\zeta)(1-\rho^2) + (1-\nu_{TT})C'_0(\zeta)\rho,$$
$$\overset{4}{F} = C_4(\zeta) - (1/2)(1+\nu_{TT})C_0(\zeta)(3\rho-\rho^3). \tag{61$_4$}$$

Finally, substituting $(61)_{0,2,4}$ into $(56)_6$ and integrating, I get

$$\overset{6}{G} = A_6(\zeta) + B_6(\zeta)\rho - (1/2)\delta C_2^{(3)}(\zeta)\rho^2 + (1/8)\gamma(1+\nu_{TT})C'_0(\zeta)(6\rho^2-\rho^4), \tag{59$_6$}$$

and substituting this expression along with $(61)_{0-2}$ into $(55)_6$ and integrating, I get

$$\overset{6}{F} = C_6(\zeta) + D_6(\zeta)\rho + (1/2)\kappa[B_6(\zeta) - (1-\nu_{TT})C'_2(\zeta)]'\rho^2 - (1/6)\delta\kappa C_2^{(4)}(\zeta)\rho^3 + (1/40)\gamma\kappa(1+\nu_{TT})C''_0(\zeta)(10\rho^3-\rho^5). \tag{60$_6$}$$

The boundary condition $(57)_6$ implies

$$B_6 = b_6 + (1-\nu_{TT})C'_2(\zeta), \quad D_6 = (1/2)\delta\kappa C_2^{(4)}(\zeta) - (5/8)\gamma\kappa(1+\nu_{TT})C''_0(\zeta), \tag{62$_6$}$$

where b_6 is a constant that may be shown from (16) to produce an axial rigid body displacement and is thus discarded. The boundary condition $(58)_6$, the conservation property (22), and the relation $\kappa^{-1} = 1-\bar{\nu}$ imply

$$A_6 = (1/2)\delta C_2^{(3)}(\zeta) - (5/8)\gamma(1+\nu_{TT})C'_0(\zeta) + C'_6(\zeta), \tag{63$_6$}$$

$$\delta C_2^{(4)}(\zeta) + 3(1-\bar{\nu})(1+\nu_{TT})C_2(\zeta) = (6/5)\gamma(1+\nu_{TT})C''_0(\zeta). \tag{64$_6$}$$

Thus,

$$\overset{6}{G} = C'_6(\zeta) + (1/2)\delta C_2^{(3)}(\zeta)(1-\rho^2) + (1-\nu_{TT})C'_2(\zeta)\rho - (1/8)\gamma(1+\nu_{TT})C'_0(\zeta)(5-\rho^2)(1-\rho^2),$$

$$\overset{6}{F} = C_6(\zeta) - (1/2)(1+\nu_{TT})C_2(\zeta)(3\rho-\rho^3) - (1/40)\gamma\kappa(1+\nu_{TT})C''_0(\zeta)\rho(1-\rho^2)^2. \tag{61$_6$}$$

Two of the four independent homogeneous solutions of $(64)_4$ and $(64)_6$ grow as $\zeta \to \infty$ and must be discarded. To determine the conditions that the remaining solutions must satisfy at $\zeta = 0$, I turn to the stresses and displacements given by (14)–(17) which, in view of (54) and $(61)_{0,2,4,6}$ and with $C(\zeta) = C_0(\zeta) + \varepsilon^{1/2}C_2(\zeta)$, take the forms

$$\frac{r\sigma_z}{R\sigma_0} = (1+\nu_{TT}))[3C(\zeta)\rho+(1/10)\varepsilon^{1/2}\gamma\kappa C_0''(\zeta)(3\rho-5\rho^3)+O(\varepsilon)], \tag{65}$$

$$\frac{R^{1/2}\tau}{H^{1/2}\sigma_0} = (1+\nu_{TT})(1-\rho^2)[(3/2)C'(\zeta)+(1/40)\varepsilon^{1/2}\gamma\kappa C_0^{(3)}(\zeta)(1-5\rho^2)+O(\varepsilon)], \tag{66}$$

$$\frac{2\bar{\mu}ru_z}{H^{1/2}R^{3/2}\sigma_0} = -[C^{(3)}(\zeta)\rho+\bar{\nu}C'(\zeta)-(1/2)\varepsilon^{1/2}(\gamma/\delta)(1+\nu_{TT})C_0'(\zeta)(3\rho-\rho^3)+O(\varepsilon)], \tag{67}$$

$$\frac{2\bar{\mu}ru_r}{R^2\sigma_0} = C''(\zeta)+O(\varepsilon). \tag{68}$$

Obviously, unless the prescribed tractions and/or displacements at the end of the tube have radial variations that match the right-hand sides of (65)–(68), the shell-like solutions are incomplete and must be supplemented by boundary layer solutions with the property that derivatives with respect to ζ are large. I now consider such solutions.

4. THE WIDE BOUNDARY-LAYER SOLUTION

Guided by the analysis in Horgan and Simmonds (1991), I let

$$\zeta = \varepsilon^{1/4}\sqrt{\gamma\kappa}\alpha, \quad \Phi = M(\rho,\alpha,\varepsilon), \quad X = \varepsilon^{-1/4}\sqrt{\gamma/\kappa}[L(\alpha,\varepsilon)+\varepsilon^{1/2}N(\rho,\alpha,\varepsilon)] \tag{69}$$

so that the governing differential equations (10) and (11) take the form

$$M_{,\rho\rho}+\frac{\varepsilon M_{,\rho}}{1+\varepsilon\rho}+\frac{\varepsilon^{1/2}(1-\nu_{TT})M_{,\alpha\alpha}}{\gamma} = \frac{N_{,\rho\alpha}}{1+\varepsilon\rho}, \tag{70}$$

$$N_{,\rho\rho}-\frac{\varepsilon N_{,\rho}}{1+\varepsilon\rho}+\varepsilon^{1/2}(\delta/\gamma\kappa)N_{,\alpha\alpha} = -(1+\varepsilon\rho)M_{,\rho\alpha}-(\delta/\gamma\kappa)L'', \tag{71}$$

while the traction-free conditions on the sides of the tube, (20) and (21), read

$$M_{,\rho}(\pm 1,\alpha,\varepsilon) = 0, \tag{72}$$

$$(1\pm\varepsilon)^2 M_{,\alpha}(\pm 1,\alpha,\varepsilon) = \gamma[L(\alpha,\varepsilon)+\varepsilon^{1/2}N(\pm 1,\alpha,\varepsilon)]. \tag{73}$$

Substituting the formal asymptotic expansions

$$\begin{aligned} M(\rho,\alpha,\varepsilon) &= \overset{0}{M}(\rho,\alpha)+\varepsilon^{1/4}\overset{1}{M}(\rho,\alpha)+\cdots, \\ N(\rho,\alpha,\varepsilon) &= \overset{0}{N}(\rho,\alpha)+\varepsilon^{1/4}\overset{1}{N}(\rho,\alpha)+\cdots, \\ L(\alpha,\varepsilon) &= \overset{0}{L}(\alpha)+\varepsilon^{1/4}\overset{1}{L}(\alpha)+\cdots, \end{aligned} \tag{74}$$

into (70)–(73), I obtain an infinite sequence of boundary-value problems, the first of which is

$$\overset{0}{M}_{,\rho\rho} = \overset{0}{N}_{,\rho\alpha}, \qquad \overset{0}{N}_{,\rho\rho} = -\overset{0}{M}_{,\rho\alpha}-(\delta/\gamma\kappa)\overset{0}{L}'', \qquad (75)_0,(76)_0$$

$$\overset{0}{M}_{,\rho}(\pm 1,\alpha) = 0, \qquad \overset{0}{M}_{,\alpha}(\pm 1,\alpha) = \gamma\overset{0}{L}(\alpha). \qquad (77)_0,(78)_0$$

Furthermore, if I set $\overset{k}{M} = \overset{k}{M}^e + \overset{k}{M}^o$, the sum of functions even and odd in ρ, then the boundary conditon (73) implies

$$\overset{0}{M}^o(1,\alpha) = \overset{1}{M}^o(1,\alpha) = 0. \tag{79$_{0,1}$}$$

Eliminating $\overset{0}{N}$ between $(75)_0$ and $(76)_0$, I obtain

$$(\overset{0}{M}_{,\rho})_{,\rho\rho} + (\overset{0}{M}_{,\rho})_{,\alpha\alpha} = -(\delta/\gamma\kappa)\overset{0}{L}^{(3)}. \tag{80$_0$}$$

This equation admits solutions of the form

$$\overset{0}{M}_{,\rho} = e^{-\lambda\alpha}[A(\lambda)\cos\lambda\rho + B(\lambda)\sin\lambda\rho] - (\delta/\gamma\kappa)\overset{0}{L}'(\alpha), \tag{81$_0$}$$

where A and B are unknown constants. Further, I require that

$$\mathscr{R}\lambda > 0 \quad \text{and} \quad \lim_{\alpha\to\infty} \overset{0}{L}'(\alpha) = 0$$

to ensure decay away from the end of the tube.

The boundary condition $(77)_0$ implies that

$$\overset{0}{L}(\alpha) = -(\gamma\kappa/\delta\lambda)A(\lambda)\cos\lambda\, e^{-\lambda\alpha} \tag{82}$$

and

$$\sin\lambda = 0. \tag{83}$$

With $A = a\sec\lambda$ and $B = -k\pi\overset{0}{b}_k$, $k = 1, 2, \ldots$, it follows from $(81)_0$, (82) and (83) that

$$\overset{0}{M} = a\,e^{-\lambda\alpha}(\lambda^{-1}\sec\lambda\sin\lambda\rho - \rho) + \overset{0}{b}_k\, e^{-k\pi\alpha}\cos k\pi\rho + E(\alpha), \tag{84$_0$}$$

where E is an unknown function. From $(79)_0$ follows

$$\tan\lambda = \lambda. \tag{85}$$

The positive roots of this equation, which may be found in Table 4.19 of Abramowitz and Stegun (1964), I denote by λ_k^o, $k = 1, 2, \ldots$, the "o" standing for *odd* following the notation of Horgan and Simmonds (1991); I denote the associated values of a in $(84)_0$ by $\overset{0}{a}_k$. It now follows from the remaining face boundary condition $(78)_0$ that

$$E = \overset{0}{a}_k(\gamma^2\kappa/\delta)\, e^{-\lambda_k^o\alpha} - (-1)^k\overset{0}{b}_k\, e^{-k\pi\alpha} \tag{86}$$

and hence that

$$\overset{0}{M} = \sum_1^\infty \left\{\overset{0}{a}_k\, e^{-\lambda_k^o\alpha}[\theta_k^o(\rho) + (\lambda_k^o)^{-2}(\gamma^2\kappa/\delta)] + \overset{0}{b}_k\, e^{-k\pi\alpha}\theta_k^e(\rho)\right\}, \tag{87}$$

where

$$\theta_k^o = \csc\lambda_k^o\sin\lambda_k^o\rho - \rho \quad \text{and} \quad \theta_k^e = \cos k\pi\rho - (-1)^k \tag{88, 89}$$

are the same *asymptotic eigenfunctions* encountered by Horgan and Simmonds (1991) in their analysis of transversely isotropic elastic strips. These eigenfunctions satisfy the orthogonality condition

$$\int_{-1}^{1} \theta_k'(\rho)\theta_l'(\rho)\,d\rho = -\int_{-1}^{1} \theta_k(\rho)\theta_k''(\rho)\,d\rho = \delta_{kl}\lambda_k^2, \tag{90}$$

where $\{\lambda_k, \theta_k\}$ stands for either of the eigenpairs $\{\lambda_k^o, \theta_k^o\}$ or $\{k\pi, \theta_k^e\}$. For future use, note that

$$\int_{-1}^{1} \rho\theta_k^o(\rho)\,d\rho = -\frac{2}{3} \quad \text{and} \quad \int_{-1}^{1} \rho^3\theta_k^o(\rho)\,d\rho = -\frac{2}{5}+\frac{4}{(\lambda_k^o)^2}. \tag{91}$$

I shall call $\overset{0}{a}_k$ and $\overset{0}{b}_k$ *wide Fourier coefficients*.

From $(78)_0$ and $(87)_0$,

$$\overset{0}{L} = -\frac{\gamma\kappa}{\delta}\sum_1^\infty \frac{\overset{0}{a}_k}{\lambda_k^o} e^{-\lambda_k^o\alpha}. \tag{92}$$

From $(75)_0$, $(76)_0$, (92) and the requirement that $\overset{0}{N} \to 0$ as $\alpha \to \infty$,

$$\overset{0}{N} = \sum_1^\infty [\overset{0}{a}_k e^{-\lambda_k^o\alpha}(1/\lambda_k^o - \csc\lambda_k^o \cos\lambda_k^o\rho) + \overset{0}{b}_k e^{-k\pi\alpha}\sin k\pi\rho], \tag{93$_0$}$$

where an unknown function of integration that depends on α only has been absorbed into $\overset{2}{L}(\alpha)$. Note that

$$\int_{-1}^{\rho} \overset{0}{N}(\tilde{\rho},\alpha)\,d\tilde{\rho} = -\sum_1^\infty [(\overset{0}{a}_k/\lambda_k^o)\,e^{-\lambda_k^o\alpha}\theta_k^o(\rho) + (\overset{0}{b}_k/k\pi)\,e^{-k\pi\alpha}\theta_k^e(\rho)]. \tag{94$_0$}$$

By (70)–(74) and $(79)_{0,1}$, it is obvious that the solutions for $\overset{1}{L}$, $\overset{1}{M}$ and $\overset{1}{N}$ are identical in form to those for $\overset{0}{L}$, $\overset{0}{M}$ and $\overset{0}{N}$, save that 0s are replaced everywhere by 1s.

In general, linear combinations of the interior and wide boundary-layer solutions are *not* sufficient to meet the various combinations of end conditions encountered in Cases A–D discussed in the Introduction. I now augment these solutions.

5. THE NARROW BOUNDARY-LAYER SOLUTION

Again guided by the analysis in Horgan and Simmonds (1991), I let

$$\zeta = \varepsilon^{3/4}\sqrt{(1-\nu_{TT})(\delta/\gamma)}\beta, \quad \Phi = P(\rho,\beta,\varepsilon), \quad X = \varepsilon^{1/4}\sqrt{(1-\nu_{TT})(\gamma/\delta)}Q(\rho,\beta,\varepsilon). \tag{95}$$

Then the differential equations (10) and (11) and the traction-free conditions (20) and (21) take the forms

$$\frac{\varepsilon^{1/2}\delta}{\gamma\kappa}\left(P_{,\rho\rho}+\frac{\varepsilon P_{,\rho}}{1+\varepsilon\rho}\right)+P_{,\beta\beta} = \frac{Q_{,\rho\beta}}{1+\varepsilon\rho}, \tag{96}$$

$$\frac{\varepsilon^{1/2}(1-\nu_{TT})}{\gamma}\left(Q_{,\rho\rho}-\frac{\varepsilon Q_{,\rho}}{1+\varepsilon\rho}\right)+Q_{,\beta\beta} = -(1+\varepsilon\rho)P_{,\rho\beta}, \tag{97}$$

$$P_{,\rho}(\pm 1,\beta,\varepsilon) = 0, \qquad (1\pm\varepsilon)^2 P_{,\beta}(\pm 1,\beta,\varepsilon) = \varepsilon(1-\nu_{TT})Q(\pm 1,\beta,\varepsilon). \tag{98, 99}$$

Substituting the expansions

$$P(\rho,\beta,\varepsilon) = \overset{0}{P}(\rho,\beta)+\varepsilon^{1/4}\overset{1}{P}(\rho,\beta)+\cdots,$$

$$Q(\rho,\beta,\varepsilon) = \overset{0}{Q}(\rho,\beta)+\varepsilon^{1/4}\overset{1}{Q}(\rho,\beta)+\cdots, \tag{100}$$

into (96)–(99), I obtain an infinite sequence of boundary-value problems, the first of which is

$$\overset{0}{P}_{,\beta\beta} = \overset{0}{Q}_{,\rho\beta}, \qquad \overset{0}{Q}_{,\beta\beta} = -\overset{0}{P}_{,\rho\beta}, \qquad (101)_0,(102)_0$$

$$\overset{0}{P}_{,\rho}(\pm 1,\beta) = 0, \qquad \overset{0}{P}_{,\beta}(\pm 1,\beta) = 0. \qquad (103_0, 104)_0$$

The differential equations $(101)_0$ and $(102)_0$ admit decaying solutions either of the form

$$\overset{0}{P} = c\mathrm{e}^{-\lambda\beta}\cos\lambda\rho, \quad \overset{0}{Q} = -c\mathrm{e}^{-\lambda\beta}\sin\lambda\rho, \qquad (105)_0$$

or else of the form

$$\overset{0}{P} = d\mathrm{e}^{-\lambda\beta}\sin\lambda\rho, \quad \overset{0}{Q} = d\mathrm{e}^{-\lambda\beta}\cos\lambda\rho, \qquad (106)_0$$

where c, d and λ are unknown constants and $\mathscr{R}\lambda > 0$. These solutions can be made to satisfy either boundary conditions $(103)_0$ or $(104)_0$, *but not both*. Choosing the latter, I get solutions of the form

$$\overset{0}{P} = \sum_1^\infty [\overset{0}{c}_k\,\mathrm{e}^{-(k-1/2)\pi\beta}\cos(k-\tfrac{1}{2})\pi\rho+\overset{0}{d}_k\,\mathrm{e}^{-k\pi\beta}\sin k\pi\rho], \qquad (107)_0$$

$$\overset{0}{Q} = \sum_1^\infty [-\overset{0}{c}_k\,\mathrm{e}^{-(k-1/2)\pi\beta}\sin(k-\tfrac{1}{2})\pi\rho+\overset{0}{d}_k\,\mathrm{e}^{-k\pi\beta}\cos k\pi\rho], \qquad (108)_0$$

where $\overset{0}{c}_k$ and $\overset{0}{d}_k$ are unknown constants. (Note that the condition $\mathscr{R}\lambda > 0$ excludes the solution $P = \overset{0}{c}_0$, $Q = 0$.) I shall call $\overset{0}{c}_k$ and $\overset{0}{d}_k$ *narrow Fourier coefficients*.

6. THE SINUOUS BOUNDARY-LAYER SOLUTION

To obtain another type of narrow boundary-layer solution that does satisfy both of the traction-free conditions on the sides of the tube, I note, just as Horgan and Simmonds (1991) showed for a strip weak in shear, that there exist *sinuous* solutions Y and Z for the tube that decay rapidly in the axial direction and oscillate even faster in the radial direction. I use these to supplement the solutions discussed in the preceding Section by setting

$$\Phi = P(\rho,\beta,\varepsilon)+\varepsilon^{1/2}Y(\eta,\beta,\varepsilon) \tag{109}$$

and

$$X = \varepsilon^{1/4}\sqrt{(1-\nu_{TT})(\gamma/\delta)}[Q(\rho,\beta,\varepsilon)+\varepsilon^{1/2}Z(\eta,\beta,\varepsilon)], \tag{110}$$

where

$$\rho = \varepsilon^{1/2}\eta. \tag{111}$$

As P and Q satisfy the differential equations (96) and (97), it follows from (10), (11), (95) and (109) that Y and Z satisfy

$$Y_{,\eta\eta}+\frac{\varepsilon^{3/2}Y_{,\eta}}{1+\varepsilon^{3/2}\eta}+\frac{\gamma\kappa\varepsilon^{1/2}Y_{,\beta\beta}}{\delta}=\frac{\gamma\kappa Z_{,\eta\beta}}{\delta(1+\varepsilon^{3/2}\eta)},\tag{112}$$

$$(1-\nu_{TT})\left(Z_{,\eta\eta}-\frac{\varepsilon^{3/2}Z_{,\eta}}{1+\varepsilon^{3/2}\eta}\right)+\varepsilon^{1/2}\gamma Z_{,\beta\beta}=-\gamma(1+\varepsilon^{3/2}\eta)Y_{,\eta\beta}.\tag{113}$$

In view of (95) and (109)–(111), the traction-free conditions (20) and (21) take the form

$$P_{,\rho}(\pm 1,\beta,\varepsilon)+Y_{,\eta}(\pm\varepsilon^{-1/2},\beta,\varepsilon)=0,\tag{114}$$

$$(1\pm\varepsilon)^2[P_{,\beta}(\pm 1,\beta,\varepsilon)+\varepsilon^{1/2}Y_{,\beta}(\pm\varepsilon^{-1/2},\beta,\varepsilon)]=\varepsilon(1-\nu_{TT})[Q(\pm 1,\beta,\varepsilon)+\varepsilon^{1/2}Z(\pm\varepsilon^{-1/2},\beta,\varepsilon)].\tag{115}$$

Inserting the expansions

$$\begin{aligned}Y(\eta,\beta,\varepsilon)&=\overset{0}{Y}(\eta,\beta)+\varepsilon^{1/4}\overset{1}{Y}(\eta,\beta)+\cdots,\\ Z(\eta,\beta,\varepsilon)&=\overset{0}{Z}(\eta,\beta)+\varepsilon^{1/4}\overset{1}{Z}(\eta,\beta)+\cdots,\end{aligned}\tag{116}$$

into (112)–(115), I obtain an infinite sequence of boundary-value problems, the first of which reads

$$\overset{0}{Y}_{,\eta\eta}=(\gamma\kappa/\delta)\overset{0}{Z}_{,\eta\beta},\qquad (1-\nu_{TT})\overset{0}{Z}_{,\eta\eta}=-\gamma\overset{0}{Y}_{,\eta\beta},\tag{117$_0$)(118$_0$}$$

$$\overset{0}{Y}_{,\eta}(\pm\varepsilon^{-1/2},\beta)=-\overset{0}{P}_{,\rho}(\pm 1,\beta)=\sum_1^{\infty}(-1)^{k+1}[\pm(k-\tfrac{1}{2})\pi\overset{0}{c}_k\,e^{-(k-1/2)\pi\beta}+k\pi\overset{0}{d}_k\,e^{-k\pi\beta}],\tag{119$_0$}$$

$$\overset{0}{P}_{,\beta}(\pm 1,\beta)=0.\tag{120$_0$}$$

To obtain the right-hand side of $(119)_0$, I have used $(107)_0$ and assumed that term-by-term differentiation of the infinite series is legitimate.

To satisfy $(119)_0$, the β-variation of $\overset{0}{Y}$ must match that of $\overset{0}{P}_{,\rho}(\pm 1,\beta)$. This leads to the following decaying solutions of $(117)_0$ and $(118)_0$:

$$\overset{0}{Y}=\sum_1^{\infty}(-1)^k\left[\frac{(k-\frac{1}{2})\pi\overset{0}{c}_k\,e^{-(k-1/2)\pi\beta}\cos(p_k\eta)}{p_k\sin(p_k\varepsilon^{-1/2})}-\frac{k\pi\overset{0}{d}_k\,e^{-k\pi\beta}\sin(q_k\eta)}{q_k\cos(q_k\varepsilon^{-1/2})}\right],\tag{121$_0$}$$

$$\overset{0}{Z}=\frac{\delta}{\gamma\kappa}\sum_1^{\infty}(-1)^k\left[\frac{\overset{0}{c}_k\,e^{-(k-1/2)\pi\beta}\sin(p_k\eta)}{\sin(p_k\varepsilon^{-1/2})}+\frac{\overset{0}{d}_k\,e^{-k\pi\beta}\cos(q_k\varepsilon^{-1/2})}{\cos(q_k\varepsilon^{-1/2})}\right],\tag{122$_0$}$$

where

$$p_k=\frac{(k-\frac{1}{2})\gamma\pi}{\sqrt{(1-\nu_{TT})(\delta/\kappa)}},\quad q_k=\frac{k\gamma\pi}{\sqrt{(1-\nu_{TT})(\delta/\kappa)}}.\tag{123}$$

7. ASSEMBLING THE PIECES

I now show that each of the four sets of end conditions labelled A–D in the Introduction and recast as eqns (45)–(52) in Section 2 can be satisfied by assuming that

$$\begin{aligned}\Phi &= F(\rho,\zeta,\varepsilon)+\varepsilon M(\rho,\alpha,\varepsilon)+\varepsilon^{3/2}P(\rho,\beta,\varepsilon)+\varepsilon^2 Y(\eta,\beta,\varepsilon)\\ &= \overset{0}{F}(\zeta)+\varepsilon^{1/2}\overset{2}{F}(\zeta)\\ &\quad+\varepsilon\sum_0^\infty[\varepsilon^{n/4}\overset{n+4}{F}(\rho,\zeta)+\varepsilon^{n/4}\overset{n}{M}(\rho,\alpha)+\varepsilon^{1/2+n/4}\overset{n}{P}(\rho,\beta)+\varepsilon^{1+n/4}\overset{n}{Y}(\eta,\beta)],\end{aligned}\tag{124}$$

$$\begin{aligned}X &= G(\rho,\zeta,\varepsilon)+\sqrt{\gamma/\kappa}\,\varepsilon^{3/4}[L(\alpha,\varepsilon)+\varepsilon^{1/2}N(\rho,\alpha,\varepsilon)]\\ &\quad+\sqrt{(1-\nu_{TT})(\gamma/\delta)}\,\varepsilon^{7/4}[Q(\rho,\beta,\varepsilon)+\varepsilon^{1/2}Z(\eta,\beta,\varepsilon)] = \overset{0}{G}(\zeta)+\varepsilon^{1/2}\overset{2}{G}(\zeta)\\ &\quad+\sqrt{\gamma/\kappa}\,\varepsilon^{3/4}\overset{0}{L}(\alpha)+\varepsilon\sum_0^\infty\{\varepsilon^{n/4}\overset{n+4}{G}(\rho,\zeta)+\sqrt{\gamma/\kappa}\,\varepsilon^{n/4}[\overset{n+1}{L}(\alpha)+\varepsilon^{1/4}\overset{n}{N}(\rho,\alpha)]\\ &\quad+\sqrt{(1-\nu_{TT})(\gamma/\delta)}\,\varepsilon^{(n+3)/4}[\overset{n}{Q}(\rho,\beta)+\varepsilon^{1/2}\overset{n}{Z}(\eta,\beta)]\}.\end{aligned}\tag{125}$$

Case A (σ_z and τ prescribed): By the last lines of (25) and (31) and (124), the edge condition (45) implies that

$$\overset{0}{F}(0) = \hat{\Phi}_0, \qquad \overset{2}{F}(0) = 0, \tag*{(126)$_0$, (126)$_2$}$$

$$\overset{4}{F}(\rho,0)+\overset{0}{M}(\rho,0) = \hat{\Phi}_4+\hat{S}_0(\rho). \tag*{(126)$_4$}$$

By the second line of (25), the first line of (31), and by $(61)_{0,2,4}$ and $(87)_0$, the above three equations reduce to

$$2(1+\nu_{TT})C_0(0) = \hat{m}_0, \qquad C_2(0) = 0, \tag*{(127)$_0$, (127)$_2$}$$

$$\begin{aligned}&C_4(0)-(1/2)(1+\nu_{TT})C_0(0)(3\rho-\rho^3)+\sum_1^\infty\{\overset{0}{a}_k[\theta_k^c(\rho)+(\lambda_k^c)^{-2}(\gamma^2\kappa/\delta)]+\overset{0}{b}_k\theta_k^s(\rho)\}\\ &= \frac{1}{2}\left\{\hat{m}_0+\frac{1}{1+\nu_{TT}}\left[\hat{m}_4-\frac{\nu_{TT}}{2}\int_{-1}^1\rho^2\hat{s}_0(\rho)\,\mathrm{d}\rho\right]\right\}+\int_{-1}^\rho(\rho-\bar{\rho})\hat{s}_0(\bar{\rho})\,\mathrm{d}\bar{\rho}.\end{aligned}\tag*{(127)$_4$}$$

Evaluating $(127)_4$ at $\rho = \pm 1$, adding the resulting expressions, noting (29) and (30) and, from (88) and (89), that $\theta_k^c(\pm 1) = \theta_k^s(\pm 1) = 0$, I get

$$C_4(0) = \frac{1}{2(1+\nu_{TT})}\left[\hat{m}_4-\frac{\nu_{TT}}{2}\int_{-1}^1\rho^2\hat{s}_0(\rho)\,\mathrm{d}\rho\right]-\frac{\gamma^2\kappa}{\delta}\sum_1^\infty\frac{\overset{0}{a}_k}{(\lambda_k^c)^2}. \tag{128}$$

Substitution of this expression back into $(127)_4$ and use of $(127)_0$ yields

$$\sum_1^\infty[\overset{0}{a}_k\theta_k^c(\rho)+\overset{0}{b}_k\theta_k^s(\rho)] = -(\hat{m}_0/4)(\rho+1)^2(\rho-2)+\int_{-1}^\rho(\rho-\bar{\rho})\hat{s}_0(\bar{\rho})\,\mathrm{d}\bar{\rho}. \tag{129}$$

The wide Fourier coefficients $\overset{0}{a}_k$ and $\overset{0}{b}_k$ may be found easily from (129) with the aid of the orthogonality condition (90). *Note*: because the right-hand side of (129) vanishes at $\rho = \pm 1$, the infinite series on the left will converge *uniformly* so long as $\hat{s}_0(\rho)$ is piecewise differentiable on $[-1, 1]$. [See the discussion of a theorem in Courant and Hilbert (1953, p. 360) in Appendix B of Horgan and Simmonds (1991).]

Even though the computation of $C_4(0)$ takes me beyond the scope of this paper, I note that the infinite series in (128) may be expressed in closed form. To do so, I multiply both sides of (129) first by ρ and then by ρ^3 and integrate with respect to ρ from -1 to 1. Taking note of (29), (30) and (91), I obtain

$$\sum_1^\infty \overset{0}{a}_k = (3\hat{m}_0/20)-(1/4)\int_{-1}^{1}\rho^3\hat{s}_0(\rho)\,\mathrm{d}\rho \tag{130}$$

and hence

$$\sum_1^\infty \frac{\overset{0}{a}_k}{(\lambda_k^0)^2} = \frac{27\hat{m}_0}{2800}-\frac{1}{40}\int_{-1}^{1}\rho^3\hat{s}_0(\rho)\,\mathrm{d}\rho+\frac{1}{80}\int_{-1}^{1}\rho^5\hat{s}_0(\rho)\,\mathrm{d}\rho. \tag{131}$$

Turning next to the edge condition (46), I note by (124) and the last line of (38) that

$$\overset{0}{F}'(0) = \hat{\Phi}_{0\zeta}, \qquad \overset{2}{F}'(0) = 0, \qquad (132)_0,(132)_2$$

$$\sqrt{(1-\nu_{TT})(\delta/\kappa)}\,\overset{0}{M}_{,\alpha}(\rho,0)+\gamma\overset{0}{P}_{,\beta}(\rho,0) = 0. \qquad (132)_3$$

By the first line of (38), $(61)_{0,2}$ and $(107)_0$, I obtain

$$2(1+\nu_{TT})C_0'(0) = \hat{q}_0, \qquad C_2'(0) = 0, \qquad (133_0),(133_2)$$

$$\sum_1^\infty \overset{0}{c}_k(k-\tfrac{1}{2})\pi\cos(k-\tfrac{1}{2})\pi\rho+\overset{0}{d}_k k\pi\sin k\pi\rho = \sqrt{(1-\nu_{TT})(\delta/\gamma^2\kappa)}\,\overset{0}{M}_{,\alpha}(\rho,0). \qquad (133)_3$$

Observe that in Case A:

(a) $C_2(0) = C_2'(0) = 0$ means that the computation of the first asymptotic correction to classical shell theory does *not* depend on the boundary-layer solutions.

(b) the lowest-order contribution to Φ in the wide boundary layer, $\overset{0}{M}(\rho,\alpha)$, is determined *first*—in terms of the wide Fourier coefficients $\overset{0}{a}_k$ and $\overset{0}{b}_k$—from (129). With this solution in hand, the lowest-order contribution to Φ in the narrow boundary layer, $\overset{0}{P}(\rho,\beta)$, is determined—in terms of the narrow Fourier coefficients $\overset{0}{c}_k$ and $\overset{0}{d}_k$—from $(133)_3$.

Case B (σ_z and u_r prescribed): The edge condition (47) of Case B is identical to the edge condition (45) of Case A; thus $C_0(0)$ and $C_2(0)$ are given by $(127)_{0,2}$. By (40), (41), (69), (95), (124) and (125), the end condition (48) implies that

$$\overset{0}{G}'(0) = \hat{\Delta}_0, \qquad \overset{2}{G}'(0)+(1-\bar{\nu})\overset{0}{L}'(0) = 0, \qquad \overset{1}{L}'(0) = 0, \qquad (134)_0,(134)_2,(134)_3$$

$$\overset{4}{G}_{,\zeta}(\rho,0)+(1-\bar{\nu})\overset{2}{L}'(0)+(1-\bar{\nu})\overset{0}{N}_{,\alpha}(\rho,0)+(\gamma/\delta)\overset{0}{Q}_{,\beta}(\rho,0) = \hat{\Delta}_4+\hat{\Omega}_0(\rho), \qquad (134)_4$$

where I have used (9) to set $\kappa^{-1} = (1-\bar{\nu})$. Inserting $(59)_{0,2}$ and (92) into $(134)_{0,2}$ and noting (130), I get

$$C_0''(0) = \hat{\Delta}_0, \qquad (135)_0$$

$$C_2''(0) = -(\gamma/\delta)\sum_1^\infty \overset{0}{a}_k = (1/4)(\gamma/\delta)\left[\int_{-1}^{1}\rho^3\hat{s}_0(\rho)\,\mathrm{d}\rho-(3/5)\hat{m}_0\right]. \qquad (135)_2$$

To satisfy $(134)_3$, I assume that $\overset{1}{L}(\alpha) \equiv 0$ [and hence that $\overset{1}{M}(\alpha) \equiv \overset{1}{N}(\alpha) \equiv 0$]. To satisfy $(134)_4$, which allows me to determine the narrow Fourier coefficients, I note from $(87)_0$ and $(93)_0$ that $\overset{0}{N}_{,\alpha} = \overset{0}{M}_{,\rho}$ and from $(107)_0$ and $(108)_0$ that $\overset{0}{Q}_{,\beta} = -\overset{0}{P}_{,\rho}$. Then inserting $(61)_4$ into $(134)_4$ and noting $(64)_4$, $(127)_0$ and $(135)_0$, I obtain

$$C_4''(0)-(3/4)(1-\bar{\nu})\hat{m}_0(0)(1-\rho^2)+(1-\nu_{TT})\hat{\Delta}_0\rho$$
$$+(1-\bar{\nu})[\overset{2}{L}'(0)+\overset{0}{M}_{,\rho}(\rho,0)]-(\gamma/\delta)\overset{0}{P}_{,\rho}(\rho,0) = \hat{\Delta}_4+\hat{\Omega}_0(\rho). \quad (136)$$

I now integrate both sides of this expression with respect to ρ from -1 to 1. Noting from (53) that

$$\int_{-1}^{1}\hat{\Omega}_0(\rho)\,\mathrm{d}\rho = -\hat{m}_0, \quad (137)$$

from $(87)_0$ that $\overset{0}{M}(\rho,0)|_{-1}^{1} = 0$, and from $(107)_0$ that $\overset{0}{P}(\pm 1,0) = 0$, I get

$$C_4''(0) = \hat{\Delta}_4-(1-\bar{\nu})\overset{2}{L}'(0)-(\bar{\nu}/2)\hat{m}_0. \quad (138)$$

Finally, introducing (138) back into (136), integrating the resulting expression with respect to ρ from -1 to ρ, and noting (129), I obtain

$$\overset{0}{P}(\rho,0) = \sum_1^{\infty}[\overset{0}{c}_k\cos(k-\tfrac{1}{2})\pi\rho+\overset{0}{d}_k\sin k\pi\rho] = -(\delta/\gamma)\left\{\int_{-1}^{\rho}[(1-\bar{\nu})(\bar{\rho}-\rho)\hat{s}_0(\bar{\rho})\right.$$
$$\left.+\hat{\Omega}_0(\bar{\rho})]\,\mathrm{d}\bar{\rho}+(\bar{\nu}/2)\hat{m}_0(1+\rho)+(1/2)(1-\nu_{TT})\hat{\Delta}_0(1-\rho^2)\right\}. \quad (139)$$

The orthogonality of $\cos(k-\tfrac{1}{2})\pi\rho$ and $\sin k\pi\rho$ on the interval $-1 \leqslant \rho \leqslant 1$ immediately yields the Fourier coefficients $\overset{0}{c}_k$ and $\overset{0}{d}_k$. Because the right-hand side of (139) vanishes at $\rho = \pm 1$, the infinite series on the left will converge uniformly provided that $\hat{\Omega}_0'(\rho) = \hat{w}_0'(\rho)+\hat{s}_0'(\rho)$ is piecewise continuous.

Observe that in Case B:

(a) $(135)_2$ implies $C_2''(0)$ can be expressed directly in terms of the edge data because $\sum_1^{\infty}\overset{0}{a}_k$ can be summed in closed form. As mentioned in the Introduction, Gregory (in a private communication) has suggested that this result, and an analogous one in case C, can also be obtained with the aid of the Betti Reciprocity Principle.

(b) as in Case A, the lowest-order contribution to Φ in the wide boundary layer, $\overset{0}{M}(\rho,\alpha)$, is determined first (in terms of the Fourier coefficients $\overset{0}{a}_k$ and $\overset{0}{b}_k$) from (129). The lowest-order contribution to Φ in the narrow boundary-layer, $\overset{0}{P}(\rho,\beta)$, is determined independently (in terms of the Fourier coefficients $\overset{0}{c}_k$ and $\overset{0}{d}_k$) from (139).

Case C (τ and u_z prescribed): First observe that, because the edge condition (49) of case C is identical to (46) of Case A, $2(1+\nu_{TT})C_0'(0) = \hat{q}_0$ and $C_2'(0) = 0$ as in $(133)_{0,2}$.

Next, the edge condition (50) yields, by the last line of (38), (43), (124) and (125),

$$\overset{4}{G}_{,\rho}(\rho,0) = \delta\hat{\Psi}_0\rho+(1-\nu_{TT})\hat{\Phi}_{0\zeta}, \quad (140)_4$$

$$\sqrt{\gamma/\kappa}\,\overset{0}{N}_{,\rho}(\rho,0) = \delta\hat{U}_0(\rho), \quad (140)_5$$

$$\overset{6}{G}_{,\rho}(\rho,0)+\sqrt{\gamma/\kappa}\,\overset{1}{N}_{,\rho}(\rho,0) = 0. \quad (140)_6$$

In view of the first line of (38) and $(61)_4$, eqn $(140)_4$ implies $(133)_0$ and

$$C_0^{(3)}(0) = -\hat{\Psi}_0. \tag{141$_0$}$$

Noting from $(93)_0$ and (85) that $\overset{0}{N}(-1,0) = 0$, I integrate both sides of $(140)_5$ from -1 to ρ twice and use $(94)_0$ and the standard iterated integral formula to obtain

$$\sum_1^\infty [(\overset{0}{a}_k/\lambda_k^o)\theta_k^o(\rho) + (\overset{0}{b}_k/k\pi)\theta_k^e(\rho)] = \delta\sqrt{\kappa/\gamma}\int_{-1}^{\rho}(\tilde{\rho}-\rho)\hat{U}_0(\tilde{\rho})\,\mathrm{d}\tilde{\rho}, \tag{142$_0$}$$

where θ_k^o and θ_k^e are given by (88) and (89).

Finally, note that $(140)_6$ implies that $\overset{6}{G}(\rho,0)+\sqrt{\gamma/\kappa}\overset{1}{N}(\rho,0) = \text{constant}$. As $\overset{1}{N}(\rho,\alpha)$ has the same form as $\overset{0}{N}(\rho,\alpha)$, it follows from $(93)_0$ and $(94)_0$ that

$$\overset{1}{N}(\pm 1,0) = 0 \quad \text{and} \quad \int_{-1}^{1}\overset{1}{N}(\rho,0)\,\mathrm{d}\rho = 0. \tag{143}$$

Thus, $(61)_6$ and $(133)_{0,2}$ imply that

$$(1/2)\delta C_2^{(3)}(0)(1-\rho^2)-(1/16)\gamma\hat{q}_0(5-\rho^2)(1-\rho^2)+\sqrt{\gamma/\kappa}\overset{1}{N}(\rho,0) = 0. \tag{144}$$

Integrating both sides of (144) from $\rho = -1$ to $\rho = 1$ and noting (143), I conclude that

$$C_2^{(3)}(0) = (3/5)(\gamma/\delta)\hat{q}_0. \tag{145$_2$}$$

Substitution of this expression back into (144) yields

$$\overset{1}{N}(\rho,0) = (1/80)\sqrt{\gamma\kappa}\hat{q}_0(1-5\rho^2)(1-\rho^2), \tag{146}$$

from which the wide Fourier coefficients $\overset{1}{a}_k$ and $\overset{1}{b}_k$ can be determined.

Observe that in Case C:

(a) the edge conditions $(133)_2$ and $(145)_2$ needed to compute the first interior correction to classical shell theory, are expressible directly in terms of the edge datum $\hat{q}_0$—the same datum used in classical shell theory.

(b) the lowest-order contributions to the wide boundary layer, $\overset{0}{M}(\rho,\alpha)$ and $\overset{0}{N}(\rho,\alpha)$, are determined first from $(142)_0$, via the Fourier coefficients $\overset{0}{a}_k$ and $\overset{0}{b}_k$. Then the lowest-order contributions to the narrow boundary layer, $\overset{0}{P}(\rho,\beta)$ and $\overset{0}{Q}(\rho,\beta)$, are determined from $(133)_3$, via the Fourier coefficients $\overset{0}{c}_k$ and $\overset{0}{d}_k$.

Case D (u_r and u_z prescribed): By (42)–(44), (124) and (125) the edge condition (51) implies

$$\overset{4}{G}_{,\rho}(\rho,0)-(1-\nu_{TT}+\bar{\nu}\delta)\overset{0}{F}'(0) = \delta(\Lambda_0+\hat{\Psi}_0\rho), \tag{147$_4$}$$

$$\sqrt{\gamma/\kappa}\overset{0}{N}_{,\rho}(\rho,0) = \delta\hat{U}_0(\rho), \tag{147$_5$}$$

$$\overset{6}{G}_{,\rho}(\rho,0)+\sqrt{\gamma/\kappa}\overset{1}{N}_{,\rho}(\rho,0)-(1-\nu_{TT}+\bar{\nu}\delta)\overset{2}{F}'(0) = \delta\Lambda_2, \tag{147$_6$}$$

while (69), (95) and the edge condition (32) imply

$$\overset{0}{G}'(0) = \hat{\Delta}_0, \tag{148$_0$}$$

$$\overset{2}{G}'(0)+(1-\bar{\nu})\overset{0}{L}'(0) = 0, \tag{148$_2$}$$

$$\overset{1}{L}'(0) = 0 \tag{148$_3$}$$

$$\overset{4}{G}_{,\zeta}(\rho,0)+(1-\bar{\nu})\overset{2}{L}'(0)-\bar{\nu}\overset{0}{M}_{,\rho}(\rho,0)-(\gamma/\delta)\overset{0}{P}_{,\rho}(\rho,0)-\overset{4}{F}_{,\rho}(\rho,0) = \hat{\Delta}_4+\hat{W}_0(\rho), \tag{148$_4$}$$

where, in $(148)_4$ I have set $\overset{0}{M}_{,\rho} = \overset{0}{N}_{,\alpha}$ and $\overset{0}{Q}_{,\beta} = -\overset{0}{P}_{,\rho}$, as I did in case B.

Substituting $(61)_{0,4}$ into $(147)_4$, I obtain $(141)_0$ and

$$-\bar{\nu}C'_0(0) = \Lambda_0. \tag{149$_0$}$$

This last relation determines Λ_0, *not* $C'_0(0)$—see the remark after eqn (23). As $(147)_5$ is identical to eqn $(140)_5$ of case C, it is equivalent to $(142)_0$ which determines the lowest-order approximation to the wide boundary layer solution via the Fourier coefficients $\overset{0}{a}_k$ and $\overset{0}{b}_k$.

Turning to $(148)_6$, I integrate with respect to ρ and use $(61)_{2,6}$ to replace $\overset{6}{G}$ and $\overset{2}{F}$. Evaluating the resulting expression at $\rho \pm 1$, I conclude that

$$-\bar{\nu}C'_2(0) = \Lambda_2 \tag{150$_2$}$$

and

$$(1/2)\delta C_2^{(3)}(0)(1-\rho^2)-(1/8)\gamma(1+\nu_{TT})C'_0(0)(5-\rho^2)(1-\rho^2)+\sqrt{\gamma/\kappa}\overset{1}{N}_{,\rho}(\rho,0) = 0. \tag{151$_2$}$$

Integrating $(151)_2$ from $\rho = -1$ to $\rho = 1$ and noting (143), I obtain

$$C_2^{(3)}(0) = (6/5)(\gamma/\delta)(1+\nu_{TT})C'_0(0), \tag{152$_2$}$$

where, at this point, $C'_0(0)$ is unknown. Note that $(150)_2$ determines Λ_2, *not* $C'_2(0)$.

The next group of expanded edge conditions, $(149)_{0,2,3}$, is identical to eqns $(134)_{0,2,3}$ of Section B. Hence, $(135)_{0,2}$ hold, but without the extreme right-hand side of $(135)_2$, i.e. $C''_0(0) = \hat{\Delta}_0$ and

$$C''_2(0) = -(\gamma/\delta)\sum_1^{\infty}\overset{0}{a}_k, \tag{153$_2$}$$

where $\overset{0}{a}_k$ may be determined in terms of $\hat{U}_0(\rho)$ from $(143)_0$. Unfortunately, a simple, closed-form expression for $\sum_1^{\infty}\overset{0}{a}_k$ does not appear possible.

At this point I have found edge conditions, $(152)_2$ and $(153))_2$, for the first interior correction to classical shell theory and need only a condition on $C''_0(0)$. To find this, I first substitute $(61)_0$ into $(149)_0$ and obtain, as in Case B,

$$C''_0(0) = \hat{\Delta}_0. \tag{153$_0$}$$

The next expanded edge condition, $(149)_2$ can be shown to lead to $(153)_2$, so no new information is obtained; $(149)_3$ implies that $\overset{1}{L}(\rho) \equiv \overset{1}{M}(\rho,\alpha) \equiv \overset{1}{N}(\rho,\alpha) \equiv 0$.

To reduce $(149)_4$, note that $(61)_4$ and the governing differntial equation of classical shell theory, $(64)_4$, imply that

$$\overset{4}{F}_{,\rho}(\rho,0) = -(3/2)(1+\nu_{TT})C_0(0)(1-\rho^2), \quad (154)$$

$$\overset{4}{G}_{,\zeta}(\rho,0) = C_4''(0)-(3/2)(1-\bar{\nu})(1+\nu_{TT})C_0(0)(1-\rho^2)+(1-\nu_{TT})C_0''(0)\rho. \quad (155)$$

Substituting these expressions into $(149)_4$ and noting $(153)_0$, I obtain

$$C_4''(0)+(3/2)\bar{\nu}(1+\nu_{TT})C_0(0)(1-\rho^2)+(1-\bar{\nu})\overset{2}{L}'(0)$$
$$-\bar{\nu}\overset{0}{M}_{,\rho}(\rho,0)-(\gamma/\delta)\overset{0}{P}_{,\rho}(\rho,0) = \hat{\Delta}_4+\hat{W}_0(\rho)-(1-\nu_{TT})\hat{\Delta}_0\rho. \quad (156)$$

Integrating this expression from $\rho = -1$ to $\rho = 1$ and noting (19), I get

$$C_4''(0) = \hat{\Delta}_4-(1-\bar{\nu})\overset{2}{L}'(0)-\bar{\nu}(1+\nu_{TT})C_0(0), \quad (157)$$

the analogue of eqn (139) of Case B. Finally, introducing (157) back into (156) and integrating the resulting expression from -1 to ρ, I obtain

$$\overset{0}{P}(\rho,0) = \sum_1^{\infty}[\overset{0}{c}_k\cos(k-\tfrac{1}{2})\pi\rho+\overset{0}{d}_k\sin k\pi\rho]$$
$$= -(\delta/\gamma)\left\{\int_{-1}^{\rho}\hat{W}_0(\bar{\rho})\,\mathrm{d}\bar{\rho}+\bar{\nu}[\overset{0}{M}(\rho,0)-\overset{0}{M}(-1,0)]\right.$$
$$\left.+(1/2)[(1-\nu_{TT})\hat{\Delta}_0-\bar{\nu}(1+\nu_{TT})C_0(0)\rho](1-\rho^2)\right\}. \quad (158)$$

From $(134)_0$ come the wide Fourier coefficients $\overset{0}{a}_k$ and $\overset{0}{b}_k$ and hence $\overset{0}{M}$. Furthermore, as $C_0''(0) = \hat{\Delta}_0$ and $C_0^{(3)}(0) = -\hat{\Psi}_0$, I can solve the classical shell equation, $(64)_4$, subject to these end conditions to find $C_0(0)$ and $C_0'(0)$. Thus, the right-hand sides of $(152)_2$ and (158) become known and from the latter equation follow the wide Fourier coefficients $\overset{0}{c}_k$ and $\overset{0}{d}_k$.

Observe that in Case D:

(a) although $C_2^{(3)}(0)$ can be expressed (ultimately) in terms of the edge data $\hat{\Delta}_0$ and $\hat{\Phi}_0$ of classical shell theory [via $(152)_2$, after the differential equation of classical shell theory has been solved and $C_0'(0)$ computed], the expression for $C_2''(0)$ given by $(153)_2$ requires an explicit solution of the wide boundary-layer equations. This jibes with what Gregory and Wan (1992) found for the isotropic tube.
(b) as in all other cases, the wide boundary-layer contribution is determined first [via $(143)_0$] and the narrow boundary-layer contribution next [via (158)].

8. SUMMARY

The function $C_0(\zeta)$ satisfies the differential equation of classical shell theory, $(64)_4$, repeated here as

$$\delta C_0^{(4)}+3(1-\bar{\nu})(1+\nu_{TT})C_0 = 0. \quad (159)$$

The lowest-order *interior* correction to classical shell theory is represented by the function $C_2(\zeta)$ which satisfies $(64)_6$, repeated here as

$$\delta C_2^{(4)}+3(1-\bar{\nu})(1+\nu_{TT})C_2 = (6/5)\gamma(1+\nu_{TT})C_0''(\zeta). \quad (160)$$

The edge conditions satisfied by $C_0(\zeta)$ and $C_2(\zeta)$ are as follows.

In case A:

$$2(1+\nu_{TT})C_0(0) = \hat{m}_0, \quad C_2(0) = 0, \tag{161}$$

$$2(1+\nu_{TT})C_0'(0) = \hat{q}_0, \quad C_2'(0) = 0. \tag{162}$$

In Case B:

$$2(1+\nu_{TT})C_0(0) = \hat{m}_0, \quad C_2(0) = 0, \tag{163}$$

$$C_0''(0) = \hat{\Delta}_0, \quad C_2''(0) = (1/4)(\gamma/\delta)\left[\int_{-1}^{1} \rho^3 \hat{s}_0(\rho)\,d\rho - (3/5)\hat{m}_0\right]. \tag{164}$$

In Case C:

$$2(1+\nu_{TT})C_0'(0) = \hat{q}_0, \quad C_2'(0) = 0, \tag{165}$$

$$C_0^{(3)}(0) = -\hat{\Psi}_0, \quad C_2^{(3)}(0) = (3/5)(\gamma/\delta)\hat{q}_0. \tag{166}$$

In Case D:

$$C_0''(0) = \hat{\Delta}_0, \quad C_2''(0) = -(\gamma/\delta)\sum_1^\infty \overset{0}{a}_k, \quad \overset{0}{a}_k \text{ from } (142)_0, \tag{167}$$

$$C_0^{(3)}(0) = -\hat{\Psi}_0, \quad C_2^{(3)}(0) = (6/5)(\gamma/\delta)(1+\nu_{TT})C_0'(0). \tag{168}$$

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REFERENCES

Abramowitz, M. and Stegun, I. A. (1964). *Handbook of Mathematical Functions*. U.S. Government Printing Office, Washington, DC.
Courant, R. and Hilbert, D. (1953). *Method of Mathematical Physics*, Vol. 1. Interscience, New York.
Gregory, R. D. and Wan, F. Y. M., (1992). Correct asymptotic theories for axisymmetric deformations of thin and moderately thick cylindrical shells (submitted for publication).
Horgan, C. O. (1974). The axisymmetric end problem for transversely isotropic circular cylinders. *Int. J. Solids Structures* **10**, 837–852.
Horgan, C. O. and Simmonds, J. G. (1991). Asymptotic analysis of an end-loaded, transversely isotropic, elastic, semi-infinite strip weak in shear. *Int. J. Solids Structures* **27**, 1895–1914.
Warren, W. E., Roark, A. L. and Bickford, W. B. (1967). End effects in semi-infinite transversely isotropic cylinders. *AIAA Jl* **5**, 1448–1455.